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The Periodically Kicked Rotator: Recurrence and/or Energy Growth

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We explore the properties of the quantum kicked rotator, its classical equivalent being the standard map. Its behavior, as found by computer studies, depends very much on the strength of the external forcing. At low strength it is seemingly recurrent in the sense of Hogg and Huberman. However, its energy increases with time at large forcings. For quantum systems, a unitary map defines the evolution over one period of time. The spectrum of this map in an infinite space does not seem to change continuously when one approaches the ratio of the frequencies of the external and of the unperturbed system by rational approximations of the golden mean.

KEY WORDS: Chaos; quantum chaos; quantum rotator.

INTRODUCTION

Since the work of Casati *et al.*,[1] several works have been devoted to the understanding of time-dependent quantum systems beyond the usual perturbation expansions. One of their motivations was to study whether the complex dynamics of classical nonintegrable generic systems has any counterpart in quantum systems. A most popular model for this study is the rigid rotator submitted to periodic kicks. Its time-dependent Schrödinger equation (in dimensionless units) is

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial\theta^2} + kV(\theta)\,S(t)\psi$$

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where the wave function $\psi(\theta)$ is smooth and 2π periodic $[\psi(\theta + 2\pi) = \psi(\theta)]$. Here k measures the strength of the time-dependent perturbation, $V(\theta)$ the position dependence of the potential [most often $V(\theta) = \cos\theta$] and $S(t)$ is a sum of kicks:

$$S(t) = \sum_{N=-\infty}^{+\infty} \delta(t - NT)$$

with T being the time lag between two consecutive kicks.

The classical counterpart of this model, described by the so-called "standard mapping,"[2] is a well-known example of chaotic behavior. The energy of the classical kicked rotator grows with time in a chaotic way following, in the mean, a diffusion law $E \sim t$ provided the perturbation is sufficiently large.

In the quantum case several different behaviors have been isolated. The question of the convenient initial condition for observing this kind of behavior is discussed below. For the moment it is enough to say that we consider everywhere the energy growth for initial conditions of finite energy.

In the work of Casati *et al.*,[1] extended later by Izrailev and Shepelyansky,[3] the phenomenon of quantum resonance has been observed. Whenever the period of the kicks T is a rational multiple of 4π the energy grows quadratically with time: $E \sim t^2$. Whenever the resonance condition is not satisfied it is tempting to call the case nonresonant. However, various behaviors of the energy are still possible and one must be particularly careful. Working with an irrational $T/4\pi$ and with moderate perturbation, Chirikov *et al.*[4] have found numerically a saturation phenomenon. Namely, the energy starts growing with time but rapidly the growth slows down and proceeds with a much smaller rate if there is any growth at all. On the other hand Hogg and Huberman have introduced the so-called recurrence property.[5] They have shown that, in what they called the "nonresonant" case, the energy must come back arbitrarily close to its initial value during the evolution of the system. However, what Hogg and Huberman called "nonresonant" may turn out to be very restrictive (beyond excluding what is called resonant by Izrailev and Shepelyansky). The evolution from time t to time $t + T$ defines a unitary map U on the L^2 space of wave functions on $[0, 2\pi[$. The system is nonresonant, in the sense of Hogg–Huberman, if a complete and discrete set of eigenfunctions on L^2 exists such that

$$U\phi_n = e^{i\alpha_n}\phi_n, \qquad n \in \mathbb{N}, \quad \alpha_n \in \mathbb{R}$$

For instance it is not clear that such a property holds for the Hamiltonian of the kicked rotator for some irrational $T/4\pi$, although the resonant case of Izrailev–Shepelyansky is excluded because the spectrum of U is continuous if

$T/4\pi$ is rational. The numerical computations of Chirikov *et al.* for irrational $T/4\pi$ show an energy growth, as we mentioned above. Estimates of the recurrence time[6] in some realistic cases have yielded impressively large times, which allows one to conclude that the recurrence, although present is quite sensitive to the magnitude of the perturbation.

It is true that unitary operators have a mathematically well-defined spectrum, and the question "what is the topology of the spectrum of the unitarity operator at hand" is well posed. We tried to answer it by seeking regularities when approaching the golden mean for $T/4\pi$ with increasing accuracy. When one does that, the quasienergy spectrum is made of an increasing number of bands, the thickness of each band decreasing much more rapidly than the inverse number of bands. And our findings point out the extreme irregularity of this band structure when the rational approximation of the golden mean is improved. This is reminiscent of the properties of the quantum map of Berry and Hannay.[7] Owing to this irregularity, predictions about the spectrum are quite uncertain. Owing to absence of any apparent self-similarity in this spectrum, it seems reasonable to assume that this spectrum is not singular continuous. Moreover,[8] a Cantor-like structure should be expected only for a particular value of the coupling, at least by analogy with the localization problem in quasiperiodic potentials. Thus, one may expect that this spectrum is dense, at least in some part of the circle. It is certainly so for $k=0$ and almost all irrational $(T/4\pi)$ by Weyl's theorem.[10] In this case, the quasi-energies are of the form $(m^2T/4\pi)$, $m \in \mathbb{Z}$, (x) = fractional part of x. The numbers $(m^2T/4\pi)$ are uniformely distributed between 0 and 1 as m runs through $\mathbb{Z}$.

Now, the above considerations concern mainly values of $T/4\pi$ which are "good" irrationals such as the golden mean. However, recently Casati and Guarneri[9] have investigated the quasi-energy spectrum for values of $T/4\pi$ which, while irrational, are "close" to rational. They have shown that one can find a set of irrational numbers, such that the quasi-energy spectrum be continuous. So for such values of $T/4\pi$, the energy growth of the system under the external perturbation is unbounded.

The notion of spectrum and of set of eigenvalues (and eigenfunctions) are sometimes considered as synonymous, by analogy with the finite dimensional case. But it is far from obvious that, in the present case, any kind of eigenfunction exists. The existence of such eigenfunctions—even in an extended space, as L^1 instead of L^2—would imply some regularity in the spectral properties of U as $T/4\pi$ approaches the golden mean. In the absence of such regularities, we conjecture that nothing like eigenfunctions exist for (most) irrational $T/4\pi$, this being likely the "typical" situation in infinite dimension space. This is strengthened again by the consideration of the $k=0$ situation. Consider a sequence of rational approximations of the golden

mean $T/4\pi$, say, p_n/q_n. We already said that the quasi-energies are the numbers $(m^2 p_n/q_n)$, $m \in \mathbb{Z}$. One may take $p_n = 1$ if p_n and q_n are mutually prime. The rational (m^2/q_n) are $1/q_n$ times the quadratic residues of q_n: these quadratic residues are integers r, $0 \leqslant r < q_n$ in the form (m^2) modulo q_n. For instance, the quadratic residues of 13 are $\{0, 1, 3, 4, 9, 10, 12\}$. Thus each quasi-energy corresponds to the hybridation of an infinite number of eigenfunctions of the unperturbed problem labeled by integers with the same quadratic residues. These discrete quasi-energies give bands at nonzero k, as the energy of bounded electrons in isolated molecules gives, in crystals bands by hybridation. But, as one increases q_n to approach more and more closely $T/4\pi$, this hybridation process has no clear-cut limit. The number $(m^2) \bmod q_n/q_n$ become dense on $[0, 1]$ by Weyl's theorem. But those numbers that are close to an irrational ρ correspond to completely different m's as q_n changes, again due to the rather irregular character of $(m^2) \bmod q_n$.

Attempts have been made[11] to relate the property of time-dependent systems as the kicked rotator to the problem of quantum localization in one-dimensional potentials. Indeed the discrete recurrence equations [see below Eq. (7)] of the time-dependent model may be seen as some Schrödinger-like equation in a discrete space. But the quasi-energies α_n are not really energies on the half real line, because they belong to a circle. Thus the hybridation phenomena are much more important than for wavepackets with energies in the usual sense. Owing to the circle topology, the neighborhood of a given quasi-energy is much "larger" in a sense that the one of an energy. This leads one to believe that the connection between localization problems and quantum rotator is not as tight as it could appear, on the basis of formal similarities.

In the present paper we reconsider the problem of the kicked rotator from various points of view. As we will present several numerical results we will start by checking in detail the exact predictions that can be made on the resonant case. Apart from the intrinsic interest of such a study this will serve as a stringent test for our numerics. We then will analyze the nonresonant case and show how the pictures of Chirikov *et al.*, on the one hand, and Hogg–Huberman, on the other hand, can be reconciled: for k sufficiently large, the energy increases with large fluctuations, while at small k the energy fluctuates without any apparent growth almost exactly as in the recurrence picture. Motivated by recent investigations on the localization problem we have examined the spectrum of eigenvalues of V by approaching an irrational $T/4\pi$ [here the golden mean $(\sqrt{5}-1)/2$] by rational approximations of the Legendre continued fractions. We have not been able to discover any rule for extrapolating the quasi-energy spectrum, even approximately, from one Legendre approximation to the next one. In the conclusion

of this paper we recapitulate our results and present physical systems whose behavior could be described by the dynamics of the kicked rotator.

THE MODEL

Let us start with the Hamiltonian of the periodically kicked rotator:

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi}{\partial\theta^2} + kV(\theta)\,S(t)\psi \tag{1}$$

Here $S(t) = \sum_{N=-\infty}^{+\infty} \delta(t - NT)$. Introducing the dimensionless time $\tau = t/T$ we obtain

$$\frac{\partial\psi}{\partial\tau} = i\frac{T}{2}\frac{\partial^2\psi}{\partial\theta^2} - ikV(\theta)\sum_{N=-\infty}^{+\infty}\delta(\tau - N)\psi \tag{2}$$

In order to integrate Eq. (2) we distinguish a two-step evolution: between successive kicks and at the kicks. The integration between kicks can be simplified if one expands ψ on the basis of the wave functions of the static rotator: $\phi_n = \langle\theta\,|\,n\rangle = e^{im\theta}/(2\pi)^{1/2}$:

$$\psi(\theta, t) = \sum_{n=-\infty}^{\infty} a_n(t)\frac{e^{in\theta}}{(2\pi)^{1/2}} \tag{3}$$

Introducing $V_{nm} = \langle n|\,V(\theta)\,|m\rangle$ we obtain

$$\frac{\partial a_n}{\partial\tau} = -i\frac{T}{2}n^2 a_n - ikS(\tau)\sum_{m=-\infty}^{+\infty} V_{nm}a_m \tag{4}$$

By writting further $a_n = e^{-i\tau Tn^2/2}b_n$ we have

$$\frac{\partial b_n}{\partial\tau} = -ikS(\tau)\sum_{m=-\infty}^{+\infty} W_{nm}b_n \tag{5}$$

with $W_{nm} = V_{nm}\exp[i(n^2 - m^2)(\tau T/2)]$.

Between kicks $S(t) = 0$ and we have simply $\partial b_n/\partial\tau = 0$. Thus $b_n^-(N) = b_n^+(N-1)$ where N denotes the kick and $+(-)$ the moment just after (before) the kick. At the kick itself Eq. (5) can be integrated (over an infinitesimal interval which encompasses the kick) to give

$$b_n^+(N) - b_n^-(N) = -i\frac{k}{2}\left[\sum_m W_{nm}(N)\,b_m^+(N) + \sum_m W_{nm}(N)\,b_m^-(N)\right] \tag{6}$$

or equivalently,

$$b_n^+(N) + i\frac{k}{2}\sum_m W_{nm}(N)\, b_m^+(N)$$

$$= b_n^+(N-1) - i\frac{k}{2}\sum_m W_{nm}(N)\, b_m^+(N-1) \tag{7}$$

We have thus obtained a simple mapping which relates the wave-function components at two successive kicks. Dropping the superscript + and with obvious matrix notations the mapping (7) can be written

$$b(N) = \frac{I - (ik/2)\, W(N)}{I + (ik/2)\, W(N)}\, b(N-1) \tag{8}$$

We remark thus that we have an explicitly unitary scheme for the iteration of the wave function components. The matrix inversion involved can easily (and accurately) be performed using Gauss' algorithm and can be particularly simplified if one chooses the form of $V(\theta)$ so as to cancel most of the off-diagonal matrix elements. The usual choice $V(\theta) = \cos\theta$ is not the best one from this point of view, leading to complicated matrix elements. A far simpler choice is $V(\theta) = (2/k)\arctan[(k/2)\cos\theta]$ (which was also proposed by Prange *et al.*[11]) and which leads to matrix elements for the kick action of the form $V_{nm} = \delta_{n,m\pm1}$. Incidentally, for small k, the two potentials are equivalent.

As far as the numerical implementation of the iteration scheme is concerned, we must stress the fact that unitarity is not a crucial test of the accuracy of our method, being explicitly built into the algorithm. It is in fact conserved up to the accuracy of the computer arithmetic. A far more important contraint is due to the spreading of the wave function which requires the presence of several hundred terms in the expansion (3) of ψ. The space needed increases very rapidly with k, and at some cases up to 10,000 components were necessary (although because of the symmetry between a_n and a_{-n} the working space dimensions can be reduced by a factor of 2).

STUDY OF THE RESONANT CASE

In the original study of Casati *et al.* the case of the fundamental quantum resonance was studied in detail. It corresponds to a period, for the kicks, which is an integer multiple of 4π: i.e., $T = 4\pi m$. One remarks easily that in this case the phase $e^{-iTn^2/2}$ which gives the evolution of the amplitudes a_n between kicks is equal to 1. Casati *et al.* have proven that in this case the energy grows quadratically with time $E \sim \tau^2$.

Izrailev and Shepelyansky have extended this result to the case of a resonant period of the form $T = 4\pi(p/q)$ with integer p and q. In this case the phases recover the value 1 only after q kicks but this leads again to a quadratic energy growth. $E = \eta\tau^2 + O(\tau)$. They have also presented a numerical evaluation of the growth coefficient η, as well as asymptotic estimates of the latter, which, however, were not verified numerically in the small k limit.

In our study we have examined in detail the case $p/q = 8/13$. In Figs. 1, 2, and 3 we present three typical pictures for the behavior of the energy as a function of time. The values are respectively $k = 0.8$, 2.0, 4.8. One notices that for the smaller value of k the energy growth is imperceptible at the beginning. If we have stopped our evolution at $\tau \sim 100$–200 we would have obtained a picture of energy recurrence "*à la* Hogg–Huberman." On the contrary with increasing k the recurrence behavior disappears and for the larger values of k we obtain a quadratic growth with practically no wiggles. This behavior reflects the crucial dependence of the energy on k, a feature which we will encounter again in the non resonant case.

A systematic study of the energy growth as a function of k has allowed us to verify the asymptotic estimates of η, proposed by Izrailev and Shepelvyansky. For $k \ll q$ they have found that $\eta \sim k^{2q}$, while for $k \gg q$

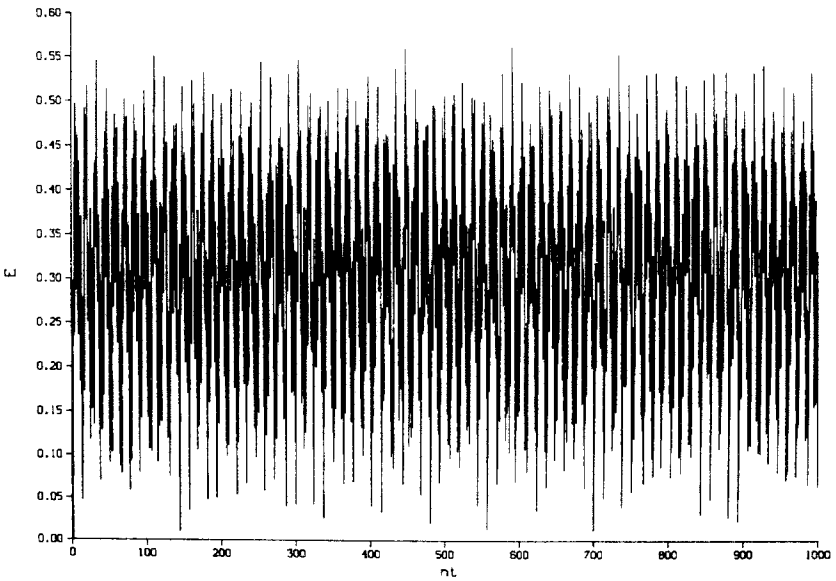

Fig. 1. Energy, E, for the rotator as a function of time (number of kicks) nt, for $T/4\pi = 8/13$ and $k = 0.8$.

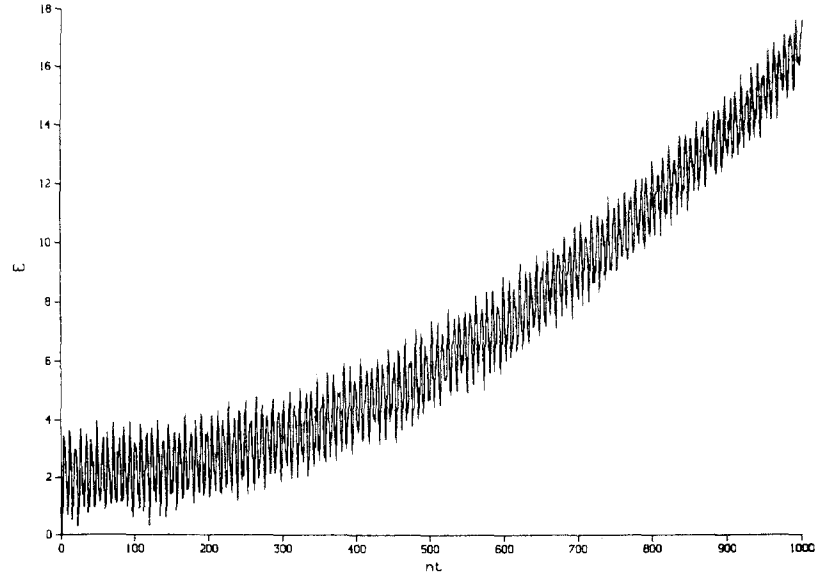

Fig. 2. Same as in Fig. 1 for $k = 2$.

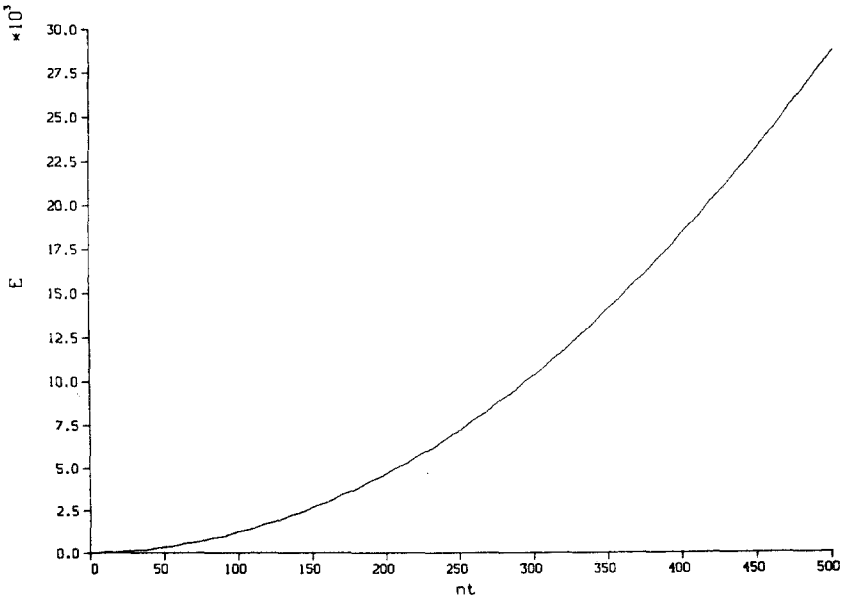

Fig. 3. Same as in Fig. 1 for $k = 4.8$.

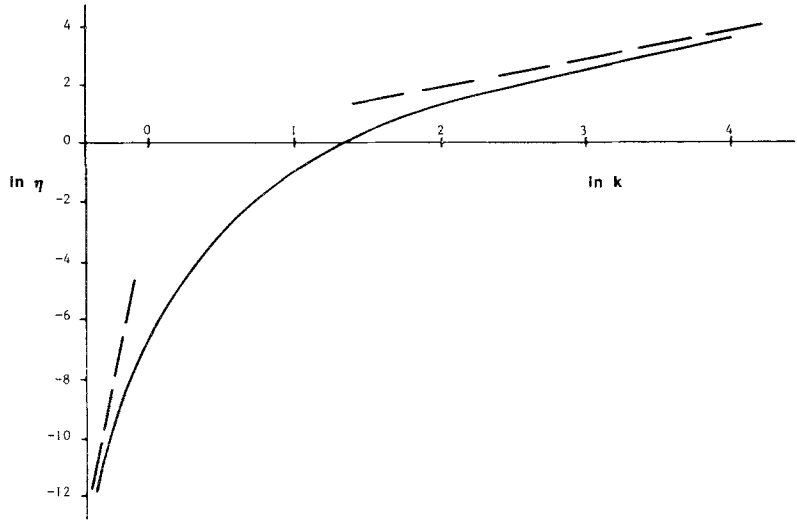

Fig. 4. Factor η of the quadratic energy growth ($E \sim \eta\tau^2$) as a function of k (in logarithmic scales) for $T/4\pi = 8/13$.

$\eta \sim k$ (Actually the aforementioned authors have obtained $\eta \sim k^2$ in the latter case, using the potential $V = k \cos\theta$. The value $\eta \sim k$ is associated to the potential $V = 2 \arctan[(k/2)\cos\theta]$ used in the present work.) In Fig. 4 we present the results for η as a function of k, over a large range of values of the latter, which allows us to reach the two asymptotic limits, materialized on the figure by the two straight-line asymptotes. One remarks readily that the agreement is quite satisfactory for large as well as for small k, i.e., for behaviors of η ranging from $\eta \underset{k\to\infty}{\sim} k$ to $\eta \underset{k\to 0}{\sim} k^{2q}$.

One remark is in order at this point. The term *quantum resonance* may be somewhat misleading as it could be interpreted as a phenomenon present only in the quantum case. This is, however, not true. Quadratic energy growth can be present in the classical case as well. However, in the latter case, it corresponds to highly unstable trajectories, and thus very difficult to study numerically.

THE NONRESONANT CASE

As explained before, this is the case where the period of the kicks is not commensurate with the natural period of the system, that is, $T/4\pi$ is irrational. Indeed this is the most general case. However, as we are interested in the limit $t \to \infty$, one could imagine that the energy $E(t, T/4\pi)$ with given

initial conditions behaves as $\varphi(t)$ for $t \to \infty$ and almost all $T/4\pi$, even though the average of $E(t, T/4\pi)$, say, $\tilde{\varphi}(t)$, over a set of values of $T/4\pi$ of finite measure at a given time is such that $\tilde{\varphi}(t) \underset{t\to\infty}{=} \varphi(t)$.

Below we shall give the numerical behavior we did find for the energy as a function of time; then we shall try to interpret our results.

NUMERICAL RESULTS: ENERGY GROWTH

Following a widely spread habit, we have chosen for $T/4\pi$ the golden mean $\sqrt{5} - 1/2$. This irrational number can be represented as $\lim_{n\to\infty} P_{n-1}/P_n$, $\{P_n\}$ being the Fibonacci with $P_0 = P_1 = 1$ and $P_{n+1} = P_n + P_{n-1}$. From the point of view of continued fraction expansion, the golden mean is, in a sense, the "most irrational" number. We then computed the energy $E(t, P_{n-1}/P_n; k)$ with the ground state $\psi(\theta) = 1/(2\pi)^{1/2}$ as initial condition in Eq. (1). For a given value of 1 and given t_0, we observed that the curve $E(t, P_{n-1}/P_n; k)$ has a rather well-defined limit as n becomes large for any t is the interval $[0, t_0]$. We took this limit value as the value of $E(t, T/4\pi; k)$ where $T/4\pi$ is the golden mean.

Using the limiting process in the approximation of the golden mean (which we recall depends on the time interval to for given k) we have

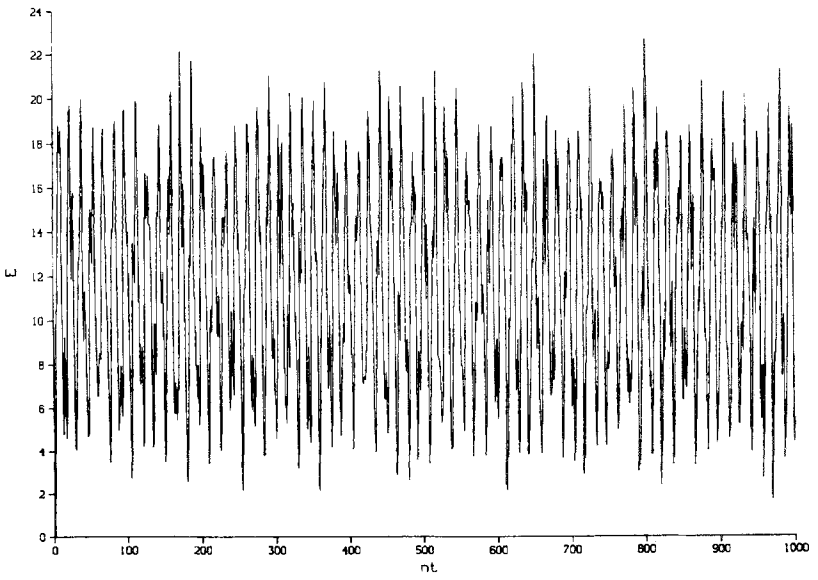

Fig. 5. Energy as a function of time for $T/4\pi$ a rational approximation of the golden mean (here $T/4\pi = 17{,}711/28{,}657$) and $k = 4$.

performed several calculations at selected values of k. For low k, typically k smaller than 10, the energy fluctuates as a function of time without any secular trend over a time interval at 1000 kicks (Fig. 5). A similar trend was observed in the case of low-order resonances. So at low k's a recurrence behavior *à la* Hogg–Huberman of the energy is observed.

At larger k the energy increases first very rapidly, roughly linearly, the initial slope increasing with k. However, this phenomenon persists only for about 50 kicks, and then the energy growth saturates. For k's of the order of ~ 100 a small energy growth accompanied by large fluctuations is observed after the initial burst (Fig. 6). At still larger values of k the relative importance of the fluctuations becomes much larger, and the secular trend of the energy toward growth, although present, is not always easily discernible (Fig. 7). A qualitatively similar behavior was observed by Chirikov *et al.*[4]

At this point we must remark that it is quite difficult to investigate numerically, with a satisfactory degree of accuracy, a law for the final growth of the energy, and it is also extremely difficult to know if there exists a sharp transition from a low-k regime, without energy growth, to a high-k regime, where the energy grows with time. Let us assume, for instance, that the energy behaves as

$$E\left(t, \frac{T}{4\pi}; k\right) \underset{t\to\infty}{\sim} t^{\mu} D(k), \qquad \mu \sim 1$$

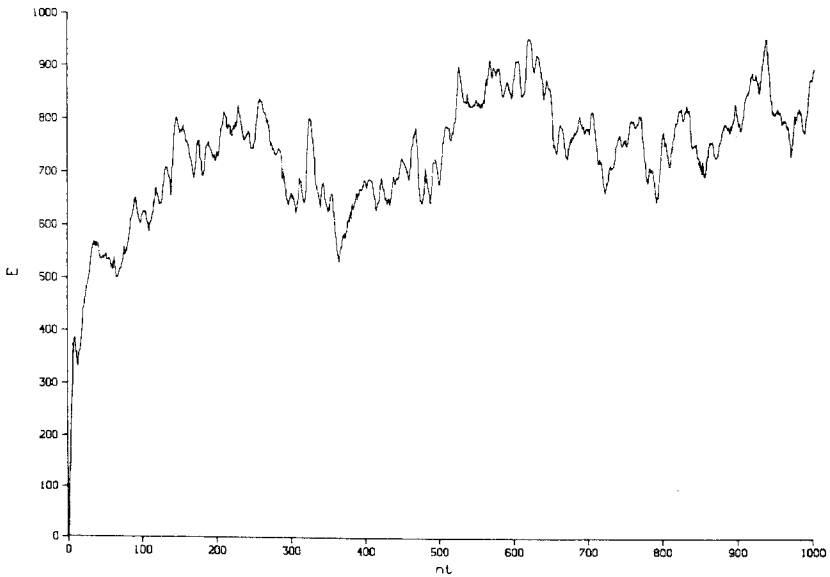

Fig. 6. Same as in Fig. 5 for $k = 80$.

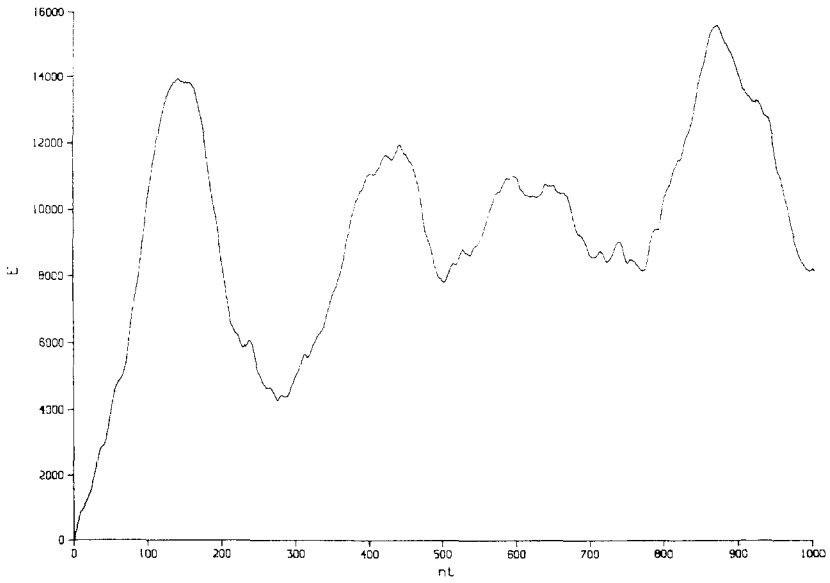

Fig. 7. Same as in Fig. 5 for $k = 160$.

The function $D(k)$, if not zero near $k = 0$, should be transcendentally small for $k \to 0+$, as, for example, $e^{-k_0^2/k^2}$, or $\exp(-k_0/k)$, because this kind of energy growth is out of reach of expansions in powers of k. Actually this kind of transcendentally small effect is quite common[12] in adiabatic theories. In the present case, one could think that, near $k = 0$, the slow time variation is due to the kicking: very approximately U is of the form $(1 + ikH)$ near $k = 0$, so that U^n differs from 1 for $kn \sim O(1)$, that is for times of order k^{-1}. Nonadiabatic phenomena, as (perhaps) the energy growth of the rotator, are typically of order $\exp(-t_{\text{fast}}/t_{\text{slow}})$, where t_{fast} is the fast time scale and t_{slow} the slow time scale, $t_{\text{fast}} \ll t_{\text{slow}}$. If one takes $t_{\text{fast}} \sim 1$ and $t_{\text{slow}} \sim 1/k$, one finds $D(k) \sim \exp(-k_0/k)$. This reasoning, unfortunately, is too rough to predict, for instance, the exponent μ. Furthermore one cannot distinguish numerically between a function growing from zero as $e^{-k_0^2/k^2}$ and a function vanishing exactly between zero and, say, $k_1 \sim 0.1k_0$ followed by a smooth behavior after k_1.

To test sensitivity of our results to the peculiarities of the model, we have performed the following "experiment." In the numerical algorithm, the unperturbed Hamiltonian occurs only through the phase factors: $\exp(im^2T/2)$. We have added random phases $\varepsilon_m (0 < \varepsilon_m < 1)$ to $m^2T/2$, destroying thus the resonance condition for rational $T/4\pi$. If one establishes the analogy with the $1 - D$ localization problem, this small amount of

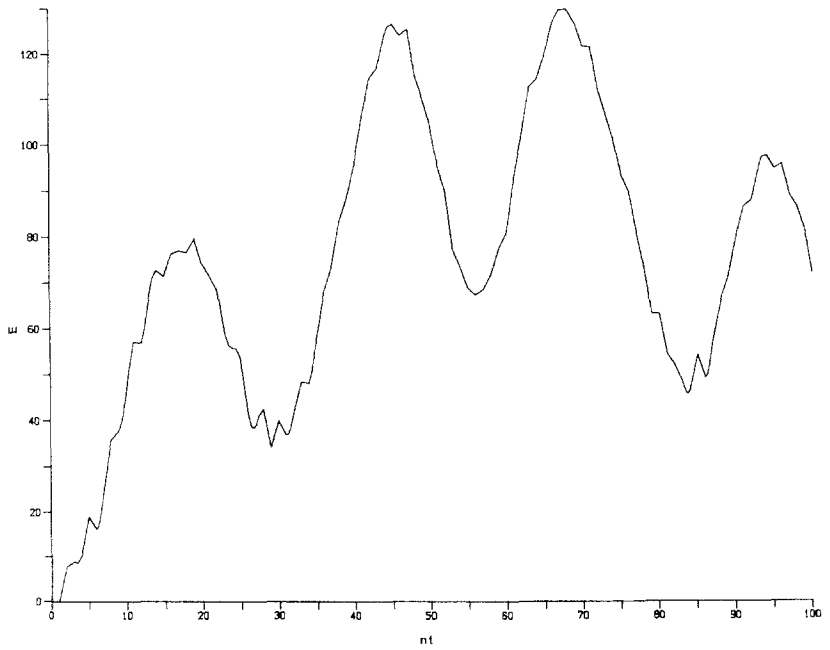

Fig. 8. Energy as a function of time for $T/4\pi$ and $k = 8$ after the addition of small random shifts to the unperturbed spectrum.

randomness is a strong perturbation indeed, as it leads to exponential localization. In the present case the addition of random ε_m's changes dramatically the pattern of energy growth in the resonant case. However, the general trends are the same as in the nonresonant case: no energy increase for low k, and increase of energy with large fluctuations for larger values of k (Fig. 8).

Thus our results agree with what was previously known. Nevertheless, we have shown that there exists a very strong sensitivity with respect to the value of k (a point that has not been raised until now). The various behaviors, recurrence or growth saturation, that were presented before, are merely manifestations of the dependence of energy on the perturbation magnitude k.

TOWARD AN INTERPRETATION OF OUR RESULTS

As explained before, it is still difficult, if not impossible to interpret our results in the light of available theories. These theories make some assumptions for the quasi-energy spectrum, that is, for the spectrum of the one step unitary operator.[13]

For the present model this spectrum is known in two limit cases. First of all if $k = 0$ (no kicks), this spectrum is simply deduced from the set of energy levels $\{E_m\}$ of the free rotator. The quasi-energies are $(m^2T/4\pi)$, where, as usual, (x) is the fractional part of x. From Weyl's theorem, the numbers $(m^2T/4\pi)$ are uniformly distributed in $[0, 1[$ as m varies in $\mathbb{N}$ for almost all $T/4\pi$. The precise diophantine condition on $(T/4\pi)$ for which this is true poses a seemingly nontrivial problem, but we can be sure that $(m^2T/4\pi)$ is not uniformly distributed if $T/4\pi$ is rational.

If $k \neq 0$, the quasi-energies are known in the resonant case. They fill continuous bands. We have already said that a possible connection has been claimed between the present problem and the quantum localization in one-dimensional quasiperiodic potentials. In this last case, it is now understood that in some cases the energy spectrum has a Cantor-like structure and this appears step by step when the irrational ratio of the two spatial frequencies is approached by rational approximations of increasing accuracy.[8] These critical values are at the transition between a discrete and a continuous spectrum, except when the irrational frequency of the quasiperiodic potential is a Liouville number.

We sought regularities in the quasi-energy spectrum by approximating the golden mean for $T/4\pi$ by the ratios of successive Fibonacci numbers. Practically, we took 2/3, 3/5, 5/8, and 8/13.

As shown in Fig. 9, the band structure of the quasi-energy spectrum at these resonances have no apparent ressemblance. In particular the band splitting leading (eventually) to a Cantor set does not show up.

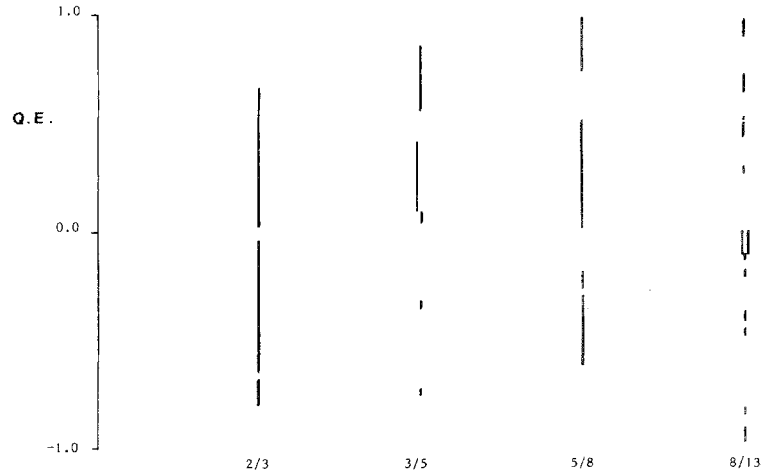

Fig. 9. Quasi-energy spectrum, in units of π, for the first small rational approximations to the golden mean. For $p/q = 5/8$ all the bands are doubly degenerate. For $p/q = 8/13$ one band is doubly degenerate.

This could be explained as follows. The bandwidth of each band is approximately of order $(k/q)^q$ for large denominators q in the rational approximation of an irrational $(T/4\pi)$.

When one considers higher and higher approximation of the golden mean with $q_n \sim (1/2)[(1+\sqrt{5})/2]^n$, the width of the bands decreases much faster than in a geometric fashion. This excludes a Cantor structure with a more or less constant reduction ratio at each step, as the one occurring in the localization problem. Moreover, as the bands have a thickness $\Delta_n \sim (k/q_n)^{q_n}$, they manifest in a sense the behavior of the system over time scales of order Δ_n^{-1}. For such times, the band structure at order $(n-1)$ is destroyed, because $\Delta_{n-1}\Delta_n^{-1} \gg 1$. This is because the bands of the quasi-energies of the matrix U^N (N positive integer) have a thickness equal to N times the thickness of those of U.

Thus, our numerical results and the previous considerations indicate that no point wise limit of the quasi-energy spectrum exists as n increases. In some sense, at each arithmetic approximation of $(T/4\pi)$, this spectrum shows a new pattern, almost independent of the spectra at lower approximations. Indeed, this leads one to think that, as n increases, there are statistical rules for generating these quasi-energy spectra. But these rules are, for the moment, unknown, and out of reach of our computational power.

The lack of regularity of the time dependence of the energy for the nonresonant case should be explained along the same lines: at time t, the system is sensitive to the structure of the quasi-energy spectrum at a rational approximation of $(T/4\pi)$ with a denominator of order t^{-1}, and this structure is very sensitive to this rational approximation.

Nevertheless, for the moment, all these considerations are very qualitative. We hope to make them more definite in the future.

CONCLUSION

Our numerical study clearly indicates the great importance of the strength of the perturbing force in these time dependent quantum problems: at low intensities, and except for resonance with small denominators, the energy seems to be recurrent, although for large intensities this has a tendency to grow with time, either as t^2 in the resonant case or more slowly in the nonresonant case.

It is of interest to speculate about the applicability of this kind of theory to experiments. Perhaps the most natural idea for this is to think at atomic or molecular physics, the external force being due to an electromagnetic wave. Unfortunately, the dimensionalizing parameter for the strength of this wave will be the electric field in a Bohr atom ($\sim$1 V over a few angströms), a huge field when compared to electric fields in electromagnetic waves. And it

should be difficult to reach experimentally a domain where nonperturbative effects (with respect to the amplitude of the external field) were important. Nevertheless the internal electric field is much lower than 1 V/1 Å in the so called Rydberg states (hydrogenic state with a huge principal quantum number), and this should be considered as a candidate for such nonperturbative effects.

One might try also to apply this kind of theory to macroscopic quantum states. To stay as close as possible to the quantum rotator, one should think at a superfluid loop in its ground state at $t = 0$ and then accelerated periodically parallel to its own plane. Indeed, this could be compared with the quantum rotator if no vortex were generated, so that the only excitation of the system were quanta of circulation around the loop.

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